PHILOSOPHY OF MATHEMATICS

A Contemporary Introduction to the World of Proofs and Pictures

Second Edition

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Chapter 1: Intro

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CHAPTER 1
Introduction: The Mathematical Image

Let's begin with a nice example, the proof that there are infinitely many prime numbers. If asked for a typical bit of real mathematics, your friendly neighbourhood mathematician is as likely to give this example as any. First, we need to know that some numbers, called 'composite', can be divided without remainder or broken into factors (e.g. $6 = 2 \times 3$, $561 = 3 \times 11 \times 17$), while other numbers, called 'prime', cannot (e.g. 2, 3, 5, 7, 11, 13, 17, . . .). Now we can ask: How many primes are there? The answer is at least as old as Euclid and is contained in the following.

> *Theorem*: There are infinitely many prime numbers.
>
> *Proof*: Suppose, contrary to the theorem, that there is only a finite number of primes. Thus, there will be a largest which we can call p. Now define a number n as 1 plus the product of all the primes:
>
> $$n = (2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p) + 1$$
>
> Is n itself prime or composite? If it is prime then our original supposition is false, since n is larger than the supposed largest prime p. So now let's consider it composite. This means that it must be divisible (without remainder) by prime numbers. However, none of the primes up to p will divide n (since we would always have remainder 1), so any number which does divide n must be greater than p. This means that there is a prime number greater than p after all. Thus, whether n is prime or composite, our supposition that there is a largest prime number is false. Therefore, the set of prime numbers is infinite.

The proof is elegant and the result profound. Still, it is typical mathematics; so, it's a good example to reflect upon. In doing so, we will begin to see the elements of *the mathematical image*, the standard conception of what mathematics is. Let's begin a list of some commonly accepted aspects. By 'commonly accepted' I mean that they would be accepted by most working mathematicians, by most educated people, and probably by most philosophers of mathematics, as well. In listing them as part of the common mathematical image we need not endorse them. Later we may even come to reject some of them – I certainly will. With this caution in mind, let's begin to outline the standard conception of mathematics.

Certainty The theorem proving the infinitude of primes seems established beyond a doubt. The natural sciences can't give us anything like this. In spite of its wonderful accomplishments, Newtonian physics has been overturned in favour of quantum mechanics and relativity. And no one today would bet too heavily on the longevity of current theories. Mathematics, by contrast, seems the one and only place where we humans can be absolutely sure we got it right.

Objectivity Whoever first thought of this theorem and its proof made a great discovery. There are other things we might be certain of, but they aren't discoveries: 'Bishops move diagonally.' This is a chess rule; it wasn't discovered; it was invented. It is certain, but its certainty stems from our resolution to play the game of chess that way. Another way of describing the situation is by saying that our theorem is an objective truth, not a convention. Yet a third way of making the same point is by saying that Martian mathematics is like ours, while their games might be quite different.

Proof is essential With a proof, the result is certain; without it, belief should be suspended. That might be putting it a bit too strongly. Sometimes mathematicians believe mathematical propositions even though they lack a proof. Perhaps we should say that without a proof a mathematical proposition is not justified and should not be used to derive other mathematical propositions. Goldbach's conjecture is an example. It says that every even number is the sum of two primes. And there is lots of evidence for it, e.g. $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 5 + 5$, $12 = 7 + 5$, and so on. It's been checked into the billions without a counter-example. Biologists don't hesitate to conclude that all ravens are black based on this sort of evidence; but mathematicians (while they might believe that Goldbach's conjecture is true) won't call it a theorem and won't use it to establish other theorems – not without a proof.

Let's look at a second example, another classic, the Pythagorean theorem. The proof below is modern, not Euclid's.

> *Theorem*: In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares on the other two sides.

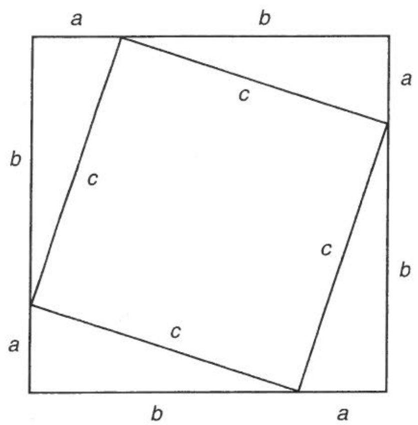

Figure 1.1

Proof: Consider two square figures, the smaller placed in the larger, making four copies of a right-angled triangle $\triangle abc$ (Figure 1.1). We want to prove that $c^2 = a^2 + b^2$.

The area of the outer square $= (a + b)^2 = c^2 + 4 \times$ (area of $\triangle abc$) $= c^2 + 2ab$, since the area of each copy of $\triangle abc$ is $\frac{1}{2}\,ab$. From algebra we have $(a + b)^2 = a^2 + 2ab + b^2$. Subtracting $2ab$ from each, we conclude $c^2 = a^2 + b^2$.

This brings out another feature of the received view of mathematics.

Diagrams There are no illustrations or pictures in the proofs of most theorems. In some there are, but these are merely a psychological aide. The diagram helps us to understand the theorem and to follow the proof – nothing more. The proof of the Pythagorean theorem or any other is the verbal/symbolic argument. Pictures can never play the role of a real proof.

Remember, in saying this I'm not endorsing these elements of the mathematical image, but merely exhibiting them. Some of these I think right, others, including this one about pictures, quite wrong. Readers might like to form their own tentative opinions as we look at these examples.

Misleading diagrams Pictures, at best, are mere psychological aids; at worst they mislead us – often badly. Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

which we can illustrate with a picture (Figure 1.2):

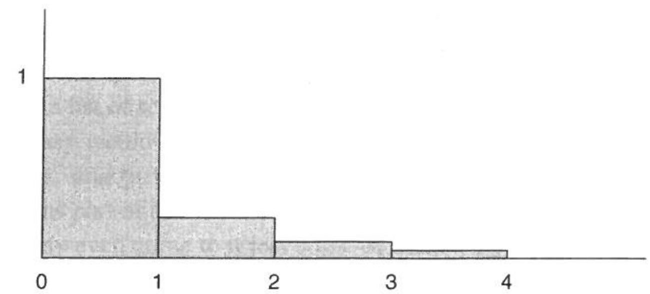

Figure 1.2 Shaded blocks correspond to terms in the series

The sum of this series is $\pi^2/6 = 1.6449\ldots$ In the picture, the sum is equal to the shaded area. Let's suppose we paint the area and that this takes *one* can of paint.

Next consider the so-called harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Here's the corresponding picture (Figure 1.3):

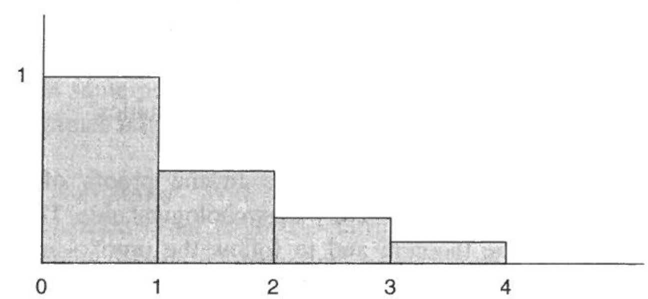

Figure 1.3

The steps keep getting smaller and smaller, just as in the earlier case, though not quite so fast. How big is the shaded area? Or rather, how much paint will be required to cover the shaded area? Comparing the two pictures, one would be tempted to say that it should require only slightly more – perhaps two or three cans of paint at most. Alas, such a guess couldn't be further off the mark. In fact, there isn't enough paint in the entire universe to cover the shaded area – it's infinite. The proof goes as follows. As we write out the series, we can group the terms:

$$\underbrace{\frac{1}{1}} + \underbrace{\frac{1}{2} + \frac{1}{3}} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}} + \underbrace{\frac{1}{8} + \frac{1}{9} + \ldots}$$

The size of the first group is obviously 1. In the second group the terms are between $\frac{1}{2}$ and $\frac{1}{4}$, so the size is between $2 \times \frac{1}{2}$ and $2 \times \frac{1}{4}$, that is, between $\frac{1}{2}$ and 1. In the next grouping of four, all terms are bigger than 1/8, so the sum is again between $\frac{1}{2}$ and 1. The same holds for the next group of 8 terms; it, too, has a sum between $\frac{1}{2}$ and 1. Clearly, there are infinitely many such groupings, each with a sum greater than $\frac{1}{2}$. When we add them all together, the total size is infinite. It would take more paint than the universe contains to cover it all. Yet, the picture doesn't give us an inkling of this startling result.

One of the most famous results of antiquity still amazes; it is the proof of the irrationality of the square root of 2. A rational number is a ratio, a fraction, such as 3/4 or 6937/528, which is composed of whole numbers. $\sqrt{9} = 3$ is rational and so is $\sqrt{(9/16)} = 3/4$; but $\sqrt{2}$ is not rational as the following theorem shows.

Theorem: The square root of 2 is not a rational number.

Proof: Suppose, contrary to the theorem, that $\sqrt{2}$ is rational, i.e. suppose that there are integers p and q such that $\sqrt{2} = p/q$. Or equivalently, $2 = (p/q)^2 = p^2/q^2$. Let us further assume that p/q is in lowest terms. (Note that $3/4 = 9/12 = 21/28$, but only the first expression is in lowest terms.)

Rearranging the above expression, we have $p^2 = 2q^2$. Thus, p^2 is even (because 2 is a factor of the right side). Hence, p is even (since the square of an odd number is odd). So it follows that $p = 2r$, for some number r. From this we get $2q^2 = p^2 = (2r)^2 = 4r^2$. Thus, $q^2 = 2r^2$, which implies that q^2 is even, and hence that q is even.

Now we have the result that both p and q are even, hence both divisible by 2, and so, not in lowest terms as was earlier supposed. Thus, we have arrived at the absurdity that p/q both *is* and *is not* in lowest terms. Therefore, our initial assumption that $\sqrt{2}$ is a rational number is false.

Classical logic Notice the structure of the proof of the irrationality of $\sqrt{2}$. We made a supposition. We derived a contradiction from this, showing the supposition is false. Then we concluded that the negation of the supposition is true. The logical principles behind this are: first, no proposition is both true and false (non-contradiction) and second, if a proposition is false, then its negation is true (excluded middle). Classical logic is a working tool of mathematics. Without this tool, much of traditional mathematics would crumble.

Strictly speaking, the proof of the irrationality of $\sqrt{2}$ is acceptable to constructive mathematicians, even though they deny the general legitimacy of classical logic. The issue will come up in more detail in a later chapter. The proof just given nicely illustrates *reduction ad absurdum* reasoning. It is also one of the all time great results, which everyone should know as a matter of general culture, just as everyone should know *Hamlet*. This is my excuse for using an imperfect example to make the point about classical logic.

Sense experience All measurement in the physical world works perfectly well with rational numbers. Letting the standard metre stick be our unit, we can measure any length with whatever desired accuracy our technical abilities will allow; but the accuracy will always be to some rational number (some fraction of a metre). In other words, we could not discover irrational numbers or incommensurable segments (i.e. lengths which are not ratios of integers) by physical measurement. It is sometimes said that we learn $2 + 2 = 4$ by counting apples and the like. Perhaps experience plays a role in grasping the elements of the natural numbers. But the discovery of the irrationality of $\sqrt{2}$ was an intellectual achievement, not at all connected to sense experience.

Cumulative history The natural sciences have revolutions. Cherished beliefs get tossed out. But a mathematical result, once proven, lasts forever. There are mathematical revolutions in the sense of spectacular results which yield new methods to work with and which focus attention in a new field – but no theorem is ever overturned. The mathematical examples I have so far discussed all predate Ptolemaic astronomy, Newtonian mechanics, Christianity and capitalism; and no doubt they will outlive them all. They are permanent additions to humanity's collection of glorious accomplishments.

Computer proofs Computers have recently played a dramatic role in mathematics. One of the most celebrated results has to do with map colouring. How many colours are needed to insure that no adjacent countries are the same colour?

> *Theorem*: Every map is four-colourable.

I won't even try to sketch the proof of this theorem. Suffice it to say that a computer was set the task of checking a very large number of cases. After a great many hours of work, it concluded that there are no counter-examples to the theorem: every map can be coloured with four colours. Thus, the theorem was established.

It's commonplace to use a hand calculator to do grades or determine our finances. We could do any of these by hand. The little gadget is a big time saver and often vastly more accurate than our efforts. Otherwise, there's really nothing new going on. Similarly, when a supercomputer tackles a big problem and spends hours on its solution, there is nothing new going on there either. Computers do what we do, only faster and perhaps more accurately. Mathematics hasn't changed because of the introduction of computers. A proof is still a proof, and that's the one and only thing that matters.

Solving problems There are lots of things we might ask, but have little chance of answering: 'Does God exist?' 'Who makes the best pizza?' These seem perfectly meaningful questions, but the chances of finding answers seems hopeless.

By contrast, it seems that every mathematical question can be answered and every problem solved. Is every even number (greater than 2) equal to the sum of two primes? We don't know now, but that's because we've been too stupid so far. Yet we are not condemned to ignorance about Goldbach's conjecture the way we are about the home of the best pizza. It's the sort of question that we should be able to answer, and in the long run we will.

Having said this, a major qualification is in order. In fact, we may have to withdraw the claim. So far, in listing the elements of the mathematical image we've made no distinction among mathematicians, philosophers and the general public. But at this point we may need to distinguish. Recent results such as Gödel's incompleteness theorem, the independence of the continuum hypothesis and others have led many mathematicians and philosophers of mathematics to believe that there are problems which are unsolvable in principle. The pessimistic principle would seem to be part of the mathematical image.

Well, enough of this. We've looked at several notions that are very widely shared and, whether we endorse them or not, they seem part of the common conception of mathematics. In sum, these are a few of the ingredients in the mathematical image:

(1) *Mathematical results are certain*
(2) *Mathematics is objective*
(3) *Proofs are essential*
(4) *Diagrams are psychologically useful, but prove nothing*
(5) *Diagrams can even be misleading*
(6) *Mathematics is wedded to classical logic*
(7) *Mathematics is independent of sense experience*
(8) *The history of mathematics is cumulative*
(9) *Computer proofs are merely long and complicated regular proofs*
(10) *Some mathematical problems are unsolvable in principle*

More could be added, but this is grist enough for our mill. Here we have the standard conception of mathematics shared by most mathematicians and non-mathematicians, including most philosophers. Yet not everyone accepts this picture. Each of these points has its several critics. Some deny that mathematics was ever certain and others say that, given the modern computer, we ought to abandon the ideal of certainty in favour of much more experimental mathematics. Some deny the objectivity of mathematics, claiming that it is a human invention after all, adding that though it's a game like chess, it is the greatest game ever played. Some deny that classical logic is indeed the right tool for mathematical inference, claiming that there are indeterminate (neither true nor false) mathematical propositions. And, finally, some would claim great virtues for pictures as proofs, far beyond their present lowly status.

We'll look at a number of issues in the philosophy of mathematics, some traditional, some current, and we'll see how much of the mathematical image endures this scrutiny. Don't be surprised should you come to abandon at least some of it. I will.

Further Reading

Many come to the philosophy of mathematics before a serious encounter with mathematics itself. If you're looking for a good place to get your feet wet, try an old classic, by Courant, Robins, and Stewart, *What is Mathematics*? If you're trying to teach yourself mathematics using standard textbooks, then I strongly urge reading popular books, as well. Rough analogies, anecdotes, and even gossip are an important part of any mathematical education. Biographies are important, too. For a collection of brief biographies of several contemporaries, try Albers and Alexanderson (eds) *Mathematical People*. There are several introductory books in the philosophy of mathematics. Shapiro, *Talking About Mathematics* is particularly nice; it covers traditional topics and Shapiro's own 'structuralism'.